Name: ____________________

Math 10550, Final Exam:
December 15, 2007

Instructor: ____________________

- Be sure that you have all 20 pages of the test.
- No calculators are to be used.
- The exam lasts for two hours.
- **When told to begin, remove this answer sheet and keep it under the rest of your test. When told to stop, hand in just this one page.**
- The Honor Code is in effect for this examination, including keeping your answer sheet under cover.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)	15.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)	16.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)	17.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)	18.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)	19.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)	20.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)	21.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)	22.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)	23.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)	24.	(a)	(b)	(c)	(d)	(e)
11.	(a)	(b)	(c)	(d)	(e)	25.	(a)	(b)	(c)	(d)	(e)
12.	(a)	(b)	(c)	(d)	(e)						
13.	(a)	(b)	(c)	(d)	(e)						
14.	(a)	(b)	(c)	(d)	(e)						

Name: ______________________________

Instructor: __________________________

Multiple Choice

1.(6 pts.) Find the limit

$$\lim_{x\to 0} \frac{1-\sqrt{x+1}}{x}.$$

(a) -1

(b) 0

(c) The limit does not exist.

(d) $\frac{1}{3}$

(e) $-\frac{1}{2}$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\sqrt{x+1}}{x} &= \lim_{x\to 0} \frac{(1-\sqrt{x+1})}{x} \cdot \frac{(1+\sqrt{x+1})}{(1+\sqrt{x+1})} \\
&= \lim_{x\to 0} \frac{1-(x+1)}{x(1+\sqrt{x+1})} \\
&= \lim_{x\to 0} \frac{-\cancel{x}}{\cancel{x}(1+\sqrt{x+1})} \\
&= \lim_{x\to 0} \frac{-1}{1+\sqrt{x+1}} \\
&= \frac{-1}{1+1} \\
&= -\frac{1}{2}
\end{aligned}$$

2.(6 pts.) The function

$$f(x) = \frac{x^2+x-6}{x^2-4}$$

has a removable discontinuity at $x = 2$. We can remove this discontinuity by defining $f(2)$ to be

(a) $\frac{1}{3}$ (b) 1 (c) 0 (d) $\frac{3}{2}$ (e) $\frac{5}{4}$

Name: ______________________________

Instructor: __________________________

Solution: Since

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} \frac{(x+3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$
$$= \frac{5}{4},$$

Thus defining $f(2) = \frac{5}{4}$ yields

$$\lim_{x \to 2} f(x) = f(2),$$

so $f(x)$ is continuous at 2.

Name: ___________________________

Instructor: _______________________

3.(6 pts.) If

$$r = \frac{\sin\theta}{1+\cos\theta},$$

then $\dfrac{dr}{d\theta} =$

(a) $\dfrac{\cos\theta + \cos^2\theta - \sin^2\theta}{(1+\cos\theta)^2}$

(b) $\dfrac{1}{1+\cos\theta}$

(c) $-\dfrac{1}{1+\cos\theta}$

(d) $\dfrac{\cos\theta}{(1+\cos\theta)^2}$

(e) $-\dfrac{\cos\theta + \cos^2\theta - \sin^2\theta}{(1+\cos\theta)^2}$

Solution:

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{(1+\cos\theta)\cos\theta - \sin\theta(-\sin\theta)}{(1+\cos\theta)^2} \\ &= \frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1+\cos\theta)^2} \\ &= \frac{1+\cos\theta}{(1+\cos\theta)^2} \\ &= \frac{1}{1+\cos\theta}\end{aligned}$$

4.(6 pts.) If

$$f(x) = \sqrt{1+\sqrt{1+x}},$$

then $f'(8) =$

(a) $\dfrac{1}{24}$ (b) $\dfrac{1}{12}$ (c) $\dfrac{1}{8}$ (d) $\dfrac{1}{9}$ (e) $\dfrac{1}{2}$

Solution: Since

$$f'(x) = \frac{1}{2}(1+\sqrt{1+x})^{-\frac{1}{2}} \cdot \frac{1}{2}(1+x)^{-\frac{1}{2}}$$

Name: ____________________________

Instructor: _________________________

plugging in $x = 8$ gives

$$\begin{aligned} f'(8) &= \frac{1}{2}(1+\sqrt{1+8})^{-\frac{1}{2}} \cdot \frac{1}{2}(1+8)^{-\frac{1}{2}} \\ &= \frac{1}{2}(1+\sqrt{9})^{-\frac{1}{2}} \cdot \frac{1}{2}(9)^{-\frac{1}{2}} \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{4}} \cdot \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{1}{24}. \end{aligned}$$

Name: ______________________________

Instructor: __________________________

5.(6 pts.) The **second** derivative of

$$y = (x+1)(x-1)(x^2+1)$$

is

(a) $24x$

(b) $x^2 + 2x - 1$

(c) $12x^2$

(d) $4x^3$

(e) $4x^2 - 2x + 1$

Solution: Since

$$y = (x+1)(x-1)(x^2+1) = (x^2-1)(x^2+1) = x^4 - 1,$$

differentiating yields

$$y' = 4x^3$$

and so

$$y'' = 12x^2.$$

6.(6 pts.) A body travels along a straight line according to the law

$$s = -t^4 - 4t^3 + 20t^2, \quad t \geq 0.$$

At what position, **after** the motion gets started, does the body first come to rest?

(a) $s = 32$ (b) $s = 36$ (c) $s = 2$

(d) $s = 12$ (e) $s = 24$

Solution: We first seek $t > 0$ such that $v(t) = 0$, where

$$v(t) = s'(t) = -4t^3 - 12t^2 + 40t.$$

Since factoring gives

$$v(t) = -4t(t^2 + 3t - 10) = -4t(t-2)(t+5),$$

we see that $t = 2$ is the first time when the body is at rest. The position at $t = 2$ is

$$s(2) = -2^4 - 4 \cdot 2^3 + 20 \cdot 2^2 = 32.$$

Name: ____________________________

Instructor: _________________________

7.(6 pts.) The equation of the tangent line to the curve

$$y = x^3 + 6x^2 + 10x + 6$$

at $x = -2$ is

(a) $y = \dfrac{x}{2}$

(b) $y = -2x - 2$

(c) $y = -\dfrac{1}{2}x + 1$

(d) $y = -x + 2$

(e) $y = -2x$

Solution: Since

$$y' = 3x^2 + 12x + 10,$$

plugging in $x = -2$ gives

$$y'(2) = 3 \cdot (-2)^2 + 12 \cdot (-2) + 10 = -2.$$

When $x = -2$, the y-coordinate on the given curve is

$$y = (-2)^3 + 6 \cdot (-2)^2 + 10 \cdot (-2) + 6 = 2.$$

Therefore the equation of the tangent is

$$\begin{aligned} & y - 2 = -2(x + 2) \\ \Longrightarrow \quad & y = -2x - 2. \end{aligned}$$

8.(6 pts.) Use the implicit differentiation to find the equation of the tangent line to the curve

$$\sqrt{5x + 9y} = 2 + xy^2 + y$$

at the point $(0, 1)$.

(a) $y = \dfrac{4}{3}x + 1$

(b) $y = -\dfrac{5}{6}x$

(c) $y = \dfrac{1}{3}x + 1$

(d) $y = -\dfrac{5}{6}x + 1$

(e) $y = \dfrac{1}{3}x$

Solution: Implicitly differentiating both sides of the given equation yields

$$\frac{1}{2}(5x + 9y)^{-1/2} \cdot (5 + 9y') = y^2 + 2xyy' + y',$$

Name: ______________________________

Instructor: __________________________

which after plugging in $x = 0$ and $y = 1$ simplifies to

$$\frac{1}{6}(5 + 9y') = 1 + y'.$$

Solving for y' gives $y' = \frac{1}{3}$. Therefore the equation of the tangent line is

$$y = \frac{1}{3}x + 1.$$

Name: ______________________________

Instructor: __________________________

9.(6 pts.) A cylinder is carved out of ice and then left in the sun to melt. If the radius decreases at a rate of 3 inches per hour and the height decreases at a rate of 6 inches per hour, how fast is the surface area of the cylinder decreasing when the cylinder is at height 5 feet and radius one foot? (Hint: 12 inches in a foot.)

Answer: The **total** surface area decreases at a rate of

(a) $\frac{3\pi}{4}$ ft^2/hr (b) $\frac{5\pi}{4}$ ft^2/hr (c) $\frac{5\pi}{2}$ ft^2/hr

(d) $\frac{9\pi}{2}$ ft^2/hr (e) 2π ft^2/hr

Solution: The surface are is given by

$$A = 2\pi r^2 + 2\pi rh.$$

Differentiating with respect to t then gives

$$\frac{dA}{dt} = 4\pi r\frac{dr}{dt} + 2\pi(h\frac{dr}{dt} + r\frac{dh}{dt}).$$

Plugging in the given values $h = 5$, $r = 1$, $\frac{dr}{dt} = \frac{1}{4}$, and $\frac{dh}{dt} = \frac{1}{2}$ yields

$$\frac{dA}{dt} = 4\pi \cdot \frac{1}{4} + 2\pi(5 \cdot \frac{1}{4} + \frac{1}{2}) = \frac{9\pi}{2}.$$

10.(6 pts.) Use linear approximation to estimate

$$\frac{1}{\sqrt{4.1}}.$$

(a) $\frac{1}{\sqrt{4.1}} \approx \frac{81}{160}$ (b) $\frac{1}{\sqrt{4.1}} \approx \frac{39}{80}$ (c) $\frac{1}{\sqrt{4.1}} \approx \frac{9}{20}$

(d) $\frac{1}{\sqrt{4.1}} \approx \frac{79}{160}$ (e) $\frac{1}{\sqrt{4.1}} \approx \frac{41}{80}$

Solution: With $f(x) = x^{-\frac{1}{2}}$, and hence $f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$, the linear approximation of f at $x = 4.1$ is

$$f(4.1) = f(4) + f'(4)(4.1 - 4) = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{8} \cdot .1 = \frac{79}{160}.$$

Name: ______________________________

Instructor: __________________________

11.(6 pts.) The maximum and minimum values of

$$f(x) = \frac{x}{x^2+1},$$

on the interval [0,2] are

(a) $M = \frac{1}{2},\ m = 0$

(b) $M = \frac{1}{2},\ m = -\frac{1}{2}$

(c) $M = 1,\ m = -\frac{3}{25}$

(d) $M = \frac{2}{5},\ m = 0$

(e) $m = 0$ is a minimum; there is no maximum.

Solution: The critical points are where

$$f'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(1+x^2)^2}$$

equals zero and the endpoints $x = 0$, $x = 2$. Since $f'(x) = 0$ if and only if $x = \pm 1$, we take $x = 1$ as our third critical points. Since $f(0) = 0$, $f(1) = \frac{1}{2}$, and $f(2) = \frac{2}{5}$, we see that $M = \frac{1}{2}$ and $m = 0$.

12.(6 pts.) Determine the number of solutions of the equation

$$x^3 - 15x + 1 = 0$$

in the interval $[-2, 2]$. The number of solutions is

(a) 2 (b) 0 (c) 1 (d) 3 (e) 4

Solution: Set $f(x) = x^3 - 15x + 1$, so that $f'(x) = 3x^2 - 15$. Since $f(-2) = 23$ and $f(2) = -21$, the intermediate value theorem guarantees that f has at least one root in $[-2, 2]$. Because $x^2 < 5$ for $x \in [-2, 2]$, it follows that $3x^2 < 15$ and hence $f'(x) = 3x^2 - 15 < 0$ for $x \in [-2, 2]$. Thus f is strictly decreasing on $[-2, 2]$ and hence cannot have more than one zero on $[-2, 2]$. Therefore f has exactly one root on $[-2, 2]$.

Name: ______________________________

Instructor: __________________________

13.(6 pts.) Consider the function

$$f(x) = \frac{x^2+3}{x-1}.$$

One of the following statements is true. Which one?

(a) The line $y = x + 1$ is a slant asymptote of f, and the line $x = 1$ is a vertical asymptote of f.

(b) f has no horizontal or slant asymptotes, and the line $x = -1$ is a vertical asymptote.

(c) The line $y = 0$ is a horizontal asymptote of f, and the line $x = -1$ is a vertical asymptote of f.

(d) The line $y = x + 2$ is a slant asymptote of f, and the line f has no vertical asymptotes.

(e) The line $y = x - 1$ is a slant asymptote of f and the line $x = 1$ is a vertical asymptote of f.

Solution: Since (as long division easily verifies)

$$\frac{x^2+3}{x-1} = x + 1 + \frac{4}{x-1},$$

the slant asymptote is $y = x + 1$. Thus there is no horizontal asymptote. Because the denominator is undefined at $x = 1$ and $x - 1$ is not a factor of the numerator, $x = 1$ is a vertical asymptote.

14.(6 pts.) Consider the function

$$f(x) = \frac{x^2+3}{x-1}.$$

One of the following statements is true. Which one?

(a) f is increasing on the interval $(-1, 3)$.

(b) f has a local minimum at $x = -1$.

(c) f is decreasing on the intervals $(-1, 1)$ and $(1, 3)$.

(d) f is increasing on the intervals $(-\infty, -1)$ and $(1, 3)$.

(e) f has a local minimum at $x = 1$.

Solution: From the previous problem, we know

$$f(x) = x + 1 + \frac{4}{x-1},$$

so that

$$f'(x) = 1 - \frac{4}{(x-1)^2}.$$

Then for $x \neq 1$

$$\begin{aligned} f'(x) > 0 &\Leftrightarrow 1 > \frac{4}{(x-1)^2} \\ &\Leftrightarrow (x-1)^2 > 4 \\ &\Leftrightarrow (x-3)(x+1) > 0 \\ &\Leftrightarrow x < -1 \text{ or } x > 3. \end{aligned}$$

Thus f is increasing on $(-\infty, -1)$ and $(3, \infty)$ and decreasing everywhere else (i.e. on $(-1, 1)$ and $(1, 3)$).

Clearly $x = 1$ is not a local minimum since f has a vertical asymptote there.

Although $f'(-1) = 0$, this is actually because of a local maximum. Indeed

$$f''(x) = \frac{8}{(x-1)^3},$$

so $f''(-1) < 0$.

Name: ______________________________

Instructor: __________________________

15.(6 pts.) Consider the function

$$f(x) = \frac{\sqrt{9x^6 - x}}{x^3 + 1}.$$

One of the following statements is true. Which one?

(a) $y = 3$ is a horizontal asymptote of f, and $y = -3$ is not a horizontal asymptote.

(b) f has no horizontal asymptotes.

(c) $y = 0$ and $y = -3$ are both horizontal asymptotes of f.

(d) $y = \pm 3$ are both horizontal asymptotes of f.

(e) $y = 0$ is a horizontal asymptote of f.

Solution: Since the problem only asks about horizontal asymptotes, we compute limits as $x \to \pm\infty$:

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} \frac{\frac{1}{\sqrt{x^6}}}{\frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\sqrt{9 - \frac{1}{x^5}}}{1 + \frac{1}{x^3}} \\
&= \frac{\sqrt{9 - 0}}{1 + 0} \\
&= 3,
\end{aligned}$$

and similarly (since $x^3 = -\sqrt{x^6}$ when $x < 0$)

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} \frac{-\frac{1}{\sqrt{x^6}}}{\frac{1}{x^3}} \\
&= -\lim_{x\to-\infty} \frac{\sqrt{9 - \frac{1}{x^5}}}{1 + \frac{1}{x^3}} \\
&= -\frac{\sqrt{9 - 0}}{1 + 0} \\
&= -3,
\end{aligned}$$

so $y = \pm 3$ are both horizontal asymptotes of f.

Name: ______________________________

Instructor: ___________________________

16.(6 pts.) The function $f(x) = (2x+1)^4 - 24x^2 + 5x$ is concave down on which of the following intervals?

(a) $(0, 1)$

(b) $(-1, \infty)$

(c) $(-\infty, -1)$

(d) $(-1, 0)$

(e) $(-\infty, 1)$

Solution: Since

$$f'(x) = 4(2x+1)^3 \cdot 2 - 48x + 5 = 8(2x+1)^3 - 48x + 5$$

and

$$f''(x) = 24(2x+1)^2 \cdot 2 - 48 = 48(\underbrace{(2x+1)^2 - 1}_{a^2-b^2}) = 48(2x)(2x+2) = 192x(x+1),$$

finding where $f''(x) < 0$ amounts to solving $x(x+1) < 0$. The curve $x(x+1)$ is an upward-opening parabola with roots at $x = 0$ and $x = -1$, and whence is negative when $-1 < x < 0$. Therefore f is concave down on $(-1, 0)$.

Name: ____________________________

Instructor: ________________________

17.(6 pts.) An open box is to be made from a square of side one by cutting four identical squares near the vertices. The box with the largest **volume** has a **height** of

(a) $\frac{1}{6}$ (b) $\frac{3}{4}$ (c) $\frac{2}{17}$

(d) $\frac{1}{2}$ (e) $\frac{1}{4}$

Solution: If the height of the box is h (which is also the side length of the cutout square), then the volume is given by

$$V = h(1-2h)^2 = h - 4h^2 + 4h^3.$$

Thus

$$V' = 1 - 8h + 12h^2 = (4h-2)(3h - \frac{1}{2}),$$

so that $V' = 0$ when $h = \frac{1}{2}$ or $h = \frac{1}{6}$.

In order to make a box, h must be in the interval $(0, 1/2)$. Because V' is an upward-opening parabola, it must switch from positive to negative at $h = \frac{1}{6}$ and be negative until $h = \frac{1}{2}$, so $h = \frac{1}{6}$ is gives a maximum on $(0, 1/2)$.

18.(6 pts.) When applying Newton's method to approximate a root of the equation $x^3 - x + 2 = 0$, with initial guess $x_1 = 1$, the value of x_2 is:

(a) 1.5 (b) 0.5 (c) 0

(d) 2 (e) 3

Solution: With $f(x) = x^3 - x + 2$, we have

$$f'(x) = 3x^2 - 1.$$

Thus

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \frac{2}{2} \\ &= 0. \end{aligned}$$

Name: ______________________________

Instructor: __________________________

19.(6 pts.) Which of the following is a Riemann sum corresponding to the integral

$$\int_2^3 x^4 dx \ \ ?$$

(a) $\frac{2}{n}\sum_{i=1}^{n}(2+\frac{i}{n})^4$ (b) $\frac{1}{n}\sum_{i=1}^{n}(2+\frac{i}{n})^4$ (c) $\frac{1}{2n}\sum_{i=1}^{n}(\frac{i}{n})^4$

(d) $\frac{2}{n}\sum_{i=1}^{n}(\frac{2+i}{n})^4$ (e) $\frac{1}{n}\sum_{i=1}^{n}(\frac{i}{n})^4$

Solution: With $f(x) = x^4$ and $\Delta x = \frac{3-2}{n} = \frac{1}{n}$, the Riemann sum in this case is

$$\sum_{i=1}^{n} f(2+i\Delta x)\Delta x = \sum_{i=1}^{n}(2+\frac{i}{n})^4\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}(2+\frac{i}{n})^4.$$

20.(6 pts.) A function $f(x)$ defined on the interval $[-1, 1]$ has an antiderivative $F(x)$. Assume that $F(-1) = 8$ and $F(1) = 7$. Which one of the statements below is true?

(a) $\int_{-1}^{1} f(x)dx = 1$.

(b) $F(x)$ is an increasing function.

(c) $f(x)$ can be an odd function.

(d) $\int_{-1}^{1} f(x)dx = 0$.

(e) $\int_{-1}^{1} f(x)dx = -1$.

Solution: If f is not assumed continuous, then f might not be integrable, so that none of the choices are correct. Thus we add the hypothesis that f is continuous.

By the fundamental theorem of calculus,

$$\int_{-1}^{1} f(x)\,dx = F(1) - F(-1) = 7 - 8 = -1.$$

Note that $f(x)$ cannot be an odd function since if it were, then

$$\int_{-1}^{1} f(x)\,dx = 0,$$

contrary to the calculation above.

Name: ____________________________

Instructor: ________________________

21.(6 pts.) Calculate the integral

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |\sin x| dx.$$

(a) π (b) 1 (c) $\frac{\pi}{2}$

(d) 2π (e) 2

Solution: Since $\sin x < 0$ only on $(\pi, 3\pi/2)$,

$$\begin{aligned}\int_{\pi/2}^{3\pi/2} |\sin x|\, dx &= \int_{\pi/2}^{\pi} \sin x\, dx + \int_{\pi}^{3\pi/2} -\sin x\, dx \\ &= -\cos x\Big|_{\pi/2}^{\pi} + \cos x\Big|_{\pi}^{3\pi/2} \\ &= 2.\end{aligned}$$

22.(6 pts.) The volume of the solid obtained by rotating the region given by $x^2 + y^2 = 1$, $x \geq 0$ and $y \geq 0$, about the line $y = -1$ is

(a) $\pi \int_0^1 (1 - x^2) dx$

(b) $\pi \int_0^1 [1 - x^2 + 2\sqrt{1 - x^2}] dx$

(c) $2\pi \int_0^1 x[1 - x^2 + 2\sqrt{1 - x^2}] dx$

(d) $2\pi \int_0^1 x\sqrt{1 - x^2} dx$

(e) $\pi \int_0^1 (1 + \sqrt{1 - x^2})^2 dx$

Name: ______________________________

Instructor: __________________________

Solution: The outer radius is $\sqrt{1-x^2}+1$ and the inner is -1, so

$$\begin{aligned} V &= \int_0^1 \pi((\text{outer radius})^2 - (\text{inner radius})^2)\,dx \\ &= \int_0^1 \pi((\sqrt{1-x^2}+1)^2 - 1^2)\,dx \\ &= \pi \int_0^1 [1 - x^2 - 2\sqrt{1-x^2}]\,dx. \end{aligned}$$

Name: ______________________________

Instructor: __________________________

23.(6 pts.) Find the volume of the solid obtained by rotating about the y-axis the region between $y = x^2$ and $y = x^4$.

(a) $\frac{\pi}{6}$ (b) π (c) $\frac{\pi}{10}$ (d) 2π (e) $\frac{\pi}{5}$

Solution: The curves intersect when $x = 0$ and $x = 1$ (and $x = -1$, but since the solid is obtained by rotating around the y-axis, this intersection point is irrelevant). Thus

$$\begin{aligned} V &= \int_0^1 2\pi x[x^2 - x^4]\,dx \\ &= 2\pi\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 \\ &= 2\pi\left[\frac{1}{4} - \frac{1}{6}\right] \\ &= \frac{\pi}{6}. \end{aligned}$$

24.(6 pts.) Find the average of $f(x) = \sin^2(x) \cdot \cos(x)$ over $[0, \frac{\pi}{2}]$.

(a) $\frac{2}{\pi}$ (b) $\frac{1}{3\pi}$ (c) $\frac{2}{3\pi}$

(d) $\frac{1}{3}$ (e) $\frac{1}{\pi}$

Solution: The average is given by

$$\begin{aligned} \frac{1}{\pi/2}\int_0^{\pi/2} \underbrace{\sin^2(x)}_{u^2}\underbrace{\cos(x)\,dx}_{du} &= \frac{2}{\pi}\int_0^1 u^2\,du \\ &= \frac{2}{3\pi}. \end{aligned}$$

Name: ____________________________

Instructor: _________________________

25.(6 pts.) A (vertical) cylindrical tank has a height 1 meter and base radius 1 meter. It is filled full with a liquid with a density 100 kg/m^3. Find the work required to empty the tank by pumping all of the liquid to the top of the tank.

(a) 500π kg-m

(b) 100π kg-m

(c) 200π kg-m

(d) 0 kg-m

(e) 50π kg-m

Solution: None of the given solutions are correct.

We consider the cylinder sliced into n slabs of equal height Δx so that the work done on the i^{th} slice is

$$\begin{aligned} W_i &= F_i x_i \\ &= (100 \times V_i \times 9.8)x_i \\ &= (100 \times \pi(1)^2 \Delta x \times o.8)x_i \\ &= (980\pi\Delta x)x_i, \end{aligned}$$

where x_i is a point in the i^{th} slab. Then the total work is

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} W_i \\ &= 980\pi \lim_{n\to\infty} \sum_{i=1}^{n} x_i \Delta x \\ &= 980\pi \int_0^1 x\,dx \\ &= 980\pi/2 \\ &= 490\pi. \end{aligned}$$

Name: ______________________________

Instructor: ANSWERS

Math 10550, Final Exam:
December 15, 2007

- Be sure that you have all 20 pages of the test.
- No calculators are to be used.
- The exam lasts for two hours.
- **When told to begin, remove this answer sheet and keep it under the rest of your test. When told to stop, hand in just this one page.**
- The Honor Code is in effect for this examination, including keeping your answer sheet under cover.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(•)	15.	(a)	(b)	(c)	(•)	(e)
2.	(a)	(b)	(c)	(d)	(•)	16.	(a)	(b)	(c)	(•)	(e)
3.	(a)	(•)	(c)	(d)	(e)	17.	(•)	(b)	(c)	(d)	(e)
4.	(•)	(b)	(c)	(d)	(e)	18.	(a)	(b)	(•)	(d)	(e)
5.	(a)	(b)	(•)	(d)	(e)	19.	(a)	(•)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)	20.	(a)	(b)	(c)	(d)	(•)
7.	(a)	(•)	(c)	(d)	(e)	21.	(a)	(b)	(c)	(d)	(•)
8.	(a)	(b)	(•)	(d)	(e)	22.	(a)	(•)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(•)	(e)	23.	(•)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(•)	(e)	24.	(a)	(b)	(•)	(d)	(e)
11.	(•)	(b)	(c)	(d)	(e)	25.	(a)	(b)	(c)	(d)	(•)
12.	(a)	(b)	(•)	(d)	(e)						
13.	(•)	(b)	(c)	(d)	(e)						
14.	(a)	(b)	(•)	(d)	(e)						